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On gravity-wave scattering by non-secular changes in depth

By JOHN MILES

Cecil H. and Ida M. Green Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, CA 92093-0225, USA

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The reflection of a straight-crested gravity wave by a non-secular perturbation $h_1(x)$ in depth relative to an otherwise flat bottom of depth h_0 is calculated through an expansion in $\varepsilon \propto h_1/h_0$. Explicit results are developed up to second order for the sinusoidal patch $h_1 = -b\sin(m\pi x/l)$, $0 < x < l$, and reduced for Bragg resonance. Trapped modes are absent at first order but appear at second order and contribute $O(\varepsilon^2)/O(\varepsilon^3)$ to the maximum (Bragg-resonant) reflection coefficient for odd/even m. A third-order approximation that includes the dominant contributions of the third-order components of the resonant peak of the reflection coefficient for large m, but neglects the trapped modes, predicts resonant peaks that agree with the values measured by Davies & Heathershaw (1984).

1. Introduction

Linear gravity waves of velocity potential ϕ, free-surface displacement ζ and frequency ω in water of ambient depth $h(\boldsymbol{x})$ are described by

$$[\phi(\boldsymbol{x}, z, t), \zeta(\boldsymbol{x}, t)] = \mathrm{Re}\{[\Phi(\boldsymbol{x}, z), \mathrm{i}(\omega/g)\,\Phi(\boldsymbol{x}, 0)]\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{1.1}$$

where the complex potential Φ satisfies

$$\nabla^2\Phi + \Phi_{zz} = 0 \quad (-h < z < 0), \tag{1.2}$$

$$\Phi_z = \kappa\Phi \quad (z = 0), \quad \Phi_z + \boldsymbol{\nabla} h\cdot\boldsymbol{\nabla}\Phi = 0 \quad (z = -h), \tag{1.3 a, b}$$

$$\boldsymbol{x} \equiv (x, y), \quad \boldsymbol{\nabla} \equiv (\partial_x, \partial_y), \quad \kappa \equiv \omega^2/g, \tag{1.4 a–c}$$

and subscripts signify partial differentiation. We seek the solution of (1.2)–(1.4) for

$$h(x) = h_0 + h_1(x) \quad (0 < x < l), \quad h_1 = 0 \quad \text{in} \quad x \leqslant 0 \quad \text{or} \quad x \geqslant l, \quad h_1/h_0 = O(\varepsilon), \tag{1.5 a–c}$$

through the expansion ($\partial_y = 0$ throughout the subsequent development)

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots, \quad \Phi_n = \Phi_n(x, z) = O(\varepsilon^n), \tag{1.6 a, b}$$

where

$$\Phi_0 = \mathrm{e}^{\mathrm{i}k_0 x}\cosh[k_0(z + h_0)] \tag{1.7}$$

describes a straight-crested incident wave in water of uniform depth h_0. The wavenumber k_0 is the positive-real root of the dispersion equation

$$k\tanh kh_0 = \kappa. \tag{1.8}$$

The solution of (1.2)–(1.7) comprises a family of trapped modes, which are evanescent for $|x| \gg h_0$, and a propagated mode of wavenumber k_0 with incident, reflected and transmitted components. The latter yield the asymptotic approximations

$$\Phi(x,z) \sim (\mathrm{e}^{\mathrm{i}k_0x} + R\,\mathrm{e}^{-\mathrm{i}k_0x})\cosh[k_0(z+h_0)] \quad (x/h_0\downarrow-\infty) \tag{1.9a}$$

and

$$\Phi(x,z) \sim T\,\mathrm{e}^{\mathrm{i}k_0x}\cosh[k_0(z+h_0)] \quad (x/h_0\uparrow\infty), \tag{1.9b}$$

wherein the reflection and transmission coefficients R and T admit the expansions

$$R = R_1 + R_2 + \cdots, \quad T = 1 + T_1 + T_2 \cdots, \quad R_n, T_n = O(\varepsilon^n). \tag{1.10a–c}$$

The first-order (truncation at $n = 1$) problem has been solved by Long (1973) for random $h_1(\boldsymbol{x})$ and by Davies & Heathershaw (1984) for arbitrary $h_1(x)$; see also Mei (1985) and Kirby (1986). Davies & Heathershaw (1984) compare their solution with measurements for a sinusoidal ripple bed and find that it is adequate for reflection coefficients smaller than about 0.5 but may overestimate reflection for Bragg resonance. They introduce an *ad hoc* 'correction' to represent higher-order effects, but this is superseded by more accurate calculations (Davies, Guazzelli & Belzons 1989; O'Hare & Davies 1992).

I consider here the construction of higher-order approximations and the contributions of the trapped modes. In §2, I obtain (through Fourier transformation) a sequential solution of the boundary-value problems for the Φ_n. In §3, I calculate the corresponding reflection and transmission coefficients to second order and then, in §4, consider the example of a sinusoidal patch, $h = h_0 - b\sin\beta x$ $(0 \leqslant x \leqslant l)$, $\beta = m\pi/l$. The second-order approximation may be inadequate for Bragg resonance $(2k_0 = \beta)$ if m is large and even (in which case R_2 proves to be approximately in quadrature, whereas R_3 is approximately in phase, with R_1), and in §5 I develop a third-order approximation to the Bragg-resonant reflection coefficient that is adequate for $mb/2h_0 = O(1)$ and agrees well with Davies & Heathershaw's (1984) measured resonant peaks for $m = 4$, 8 and 20 (2, 4 and 10 in their notation).

Nonlinearity would alter the linear approximation to R by $A \equiv 1 + O(k_0^2a^2)$, where a is the amplitude of the incident wave. This factor must be compared with $R_3/R_1 = 1 + O(\varepsilon^2)$ for the present (§5) approximation to the Bragg reflection coefficient, which therefore appears to require $(k_0a/\varepsilon)^2 \ll 1$ for its validity; however, the coefficient of $k_0^2a^2$ in A presumably is much smaller than that of ε^2 in R_3/R_1.

The present approximation, in which the expansion parameter is a measure of the variation in depth, may be compared with the mild-slope (Mei 1983, §3.5) and modified mild-slope approximations (Chamberlain & Porter 1995; Miles & Chamberlain 1998), in which the expansion parameter is a measure of the bottom slope. See O'Hare & Davies (1992) and Suh, Lee & Park (1997) for more extensive lists of references. The present approximation is somewhat simpler, and may be more efficient, than these mild-slope approximations; however, it does not accommodate secular changes in depth, is less efficient than Mei's (1985) asymptotic approximation for a long $(m \gg 1)$ ripple bed, and is less powerful than some of the more sophisticated methods cited by O'Hare & Davies (1992) and Suh *et al.* (1997).

2. Fourier-transform solution

Substituting (1.6) into (1.2), (1.3*a*) and the expansion of (1.3*b*) about $z = -h_0$, we obtain the sequence (for $n = 1, 2, \cdots$)

$$\Phi_{nxx} + \Phi_{nzz} = 0 \quad (-h_0 < z < 0), \tag{2.1}$$

$$\Phi_{nz} = \kappa\Phi_n \quad (z = 0), \quad \Phi_{nz} = -Q_{nx} \quad (z = -h_0), \tag{2.2a, b}$$

where

$$\left.\begin{aligned} Q_1 = h_1\,\Phi_{0x}, \quad Q_2 = h_1\,\Phi_{1x} - \tfrac{1}{2}h_1^2\,\Phi_{0xz}, \\ Q_3 = h_1\,\Phi_{2x} - \tfrac{1}{2}h_1^2\,\Phi_{1xz} + \tfrac{1}{6}h_1^3\,\Phi_{0xzz}, \cdots \quad (z = -h_0). \end{aligned}\right\} \tag{2.3}$$

The Φ_n then may be determined sequentially, starting from (1.7) for Φ_0.

Guided by Havelock's (1929, §5) treatment of surface-forced gravity waves, we construct the solution of the bottom-forcing problem (in which the $Q_n(x)$ may be regarded as fluid inputs) posed by (2.1) and (2.2) through the finite Fourier-transform pair

$$\Psi(x,k) = \int_{-h_0}^{0} \Phi(x,z)\cos\,[\mathrm{i}k(z+h_0)]\,\mathrm{d}z \tag{2.4a}$$

and
$$\Phi(x,z) = 2\sum_k K(k)\,\Psi(x,k)\cos[\mathrm{i}k(z+h_0)], \tag{2.4b}$$

where
$$K(k) = \frac{k^2-\kappa^2}{(k^2-\kappa^2)\,h_0+\kappa}, \tag{2.5}$$

and the summation is over the positive-real root k_0 and the infinite, discrete set of positive-imaginary roots, $k = \mathrm{i}\mathscr{k}$, of (1.8). Transforming (2.1), reducing the transform of Φ_{nzz} through integration by parts, and invoking (2.2a, b), we obtain

$$\Psi_{nxx} + k^2\,\Psi_n = -Q_n'(x), \tag{2.6}$$

the solution of which, subject to finiteness and radiation conditions for $x \rightarrow \pm\infty$, is given by (recall that $Q_n = 0$ outside of $0 < x < l$)

$$\Psi_n(x,k) = -\tfrac{1}{2}(\mathrm{i}k)^{-1}\int_0^l Q_n'(\xi)\,\mathrm{e}^{\mathrm{i}k|x-\xi|}\,\mathrm{d}\xi. \tag{2.7}$$

We cast the inverse transform of Ψ_n, as determined by (2.4b), in the Green's-function form

$$\Phi_n(x,z) = \int_0^l G(x-\xi,z)\,Q_n'(\xi)\,\mathrm{d}\xi \tag{2.8a}$$

$$= \partial_x \int_0^l G(x-\xi,z)\,Q_n(\xi)\,\mathrm{d}\xi, \tag{2.8b}$$

where (2.8b) follows from (2.8a) through integration by parts and the invocation of $Q_n(0) = Q_n(l) = 0$ (which follow from $h_1(0) = h_1(l) = 0$),

$$G(x,z) = -\sum_k (\mathrm{i}k)^{-1}\,K(k)\,\mathrm{e}^{\mathrm{i}k|x|}\cos[\mathrm{i}k(z+h_0)] \tag{2.9a}$$

$$= -(\mathrm{i}k_0)^{-1}\,K_0\,\mathrm{e}^{\mathrm{i}k_0|x|}\cosh[k_0(z+h_0)] + \sum_{\mathscr{k}} \mathscr{k}^{-1}\,K(\mathrm{i}\mathscr{k})\,\mathrm{e}^{-\mathscr{k}|x|}\cos[\mathscr{k}(z+h_0)], \tag{2.9b}$$

and $K_0 \equiv K(k_0)$. The $\mathscr{k}$ summation comprises the trapped modes, which are evanescent for $|x| \gg h_0$. The k_0 term represents the propagated mode, and the comparison of its asymptotes (for $x \rightarrow \mp\infty$) with (1.9a, b) yields

$$R_n = K_0\int_0^l Q_n(x)\,\mathrm{e}^{\mathrm{i}k_0 x}\,\mathrm{d}x, \quad T_n = -K_0\int_0^l Q_n(x)\,\mathrm{e}^{-\mathrm{i}k_0 x}\,\mathrm{d}x. \tag{2.10a, b}$$

3. Second-order results

Proceeding through the sequence developed in §2, starting from the substitution of Φ_0 (1.7) into (2.3), and setting $z = -h_0$, we obtain

$$Q_1(x) = \mathrm{i}k_0 h_1(x)\,\mathrm{e}^{\mathrm{i}k_0 x}, \tag{3.1}$$

$$R_1 = \mathrm{i}k_0 K_0 \int_0^l h_1(x)\,\mathrm{e}^{2\mathrm{i}k_0 x}\,\mathrm{d}x, \quad T_1 = -\mathrm{i}k_0 K_0 \int_0^l h_1(x)\,\mathrm{d}x, \tag{3.2a, b}$$

$$Q_2(x) = h_1(x)\,\partial_x \int_0^l G(x-\xi)\,Q_1'(\xi)\,\mathrm{d}\xi, \tag{3.3}$$

$$R_2 = (\mathrm{i}k_0)^{-1} K_0 \int_0^l Q_1(x)\,\mathrm{d}x\,\partial_x \int_0^l G(x-\xi)\,Q_1'(\xi)\,\mathrm{d}\xi \tag{3.4a}$$

$$= -(\mathrm{i}k_0)^{-1} K_0 \int_0^l\int_0^l G(x-\xi)\,Q_1'(x)\,Q_1'(\xi)\,\mathrm{d}\xi\,\mathrm{d}x \tag{3.4b}$$

$$= -2(\mathrm{i}k_0)^{-1} K_0 \int_0^l Q_1'(x)\,\mathrm{d}x \int_0^x G(x-\xi)\,Q_1'(\xi)\,\mathrm{d}\xi, \tag{3.4c}$$

$$T_2 = (\mathrm{i}k_0)^{-1} K_0 \int_0^l \bar{Q}_1(x)\,\mathrm{d}x\,\partial_x \int_0^l G(x-\xi)\,Q_1'(\xi)\,\mathrm{d}\xi \tag{3.5a}$$

$$= -(\mathrm{i}k_0)^{-1} K_0 \int_0^l\int_0^l G(x-\xi)\,\bar{Q}_1'(x)\,Q_1'(\xi)\,\mathrm{d}\xi\,\mathrm{d}x \tag{3.5b}$$

$$= -(\mathrm{i}k_0)^{-1} K_0 \int_0^l \mathrm{d}x \int_0^x G(x-\xi)\,[Q_1'(x)\,\bar{Q}_1'(\xi) + \bar{Q}_1'(x)\,Q_1'(\xi)]\,\mathrm{d}\xi, \tag{3.5c}$$

wherein $Q_1(0) = Q_1(l) = 0$ has been invoked after the partial integrations, the reduction of (3.4b) to (3.4c) follows from the identity $G(x-\xi) = G(\xi-x)$, and $\bar{Q}_1$ is the complex-conjugate of Q_1; Q_3 and R_3 are given in the Appendix.

4. Sinusoidal patch

Consider, for example,

$$h_1 = -b\sin\beta x \quad (0 < x < l), \quad \beta l = m\pi \tag{4.1a, b}$$

(m is a positive integer), the substitution of which into (3.2) and (3.4) yields

$$R_1 = \mathrm{i}\varepsilon k_0 \beta\left(\frac{\mathrm{e}^{2\mathrm{i}k_0 l}\cos\beta l - 1}{\beta^2 - 4k_0^2}\right) \tag{4.2}$$

and

$$R_2 = 2\mathrm{i}\varepsilon^2 (k_0/K_0) \sum_k K(k)\,H(\beta, k, k_0), \tag{4.3}$$

where

$$\varepsilon \equiv K_0 b = b/[h_0 + (2k_0)^{-1}\sinh 2k_0 h_0], \tag{4.4}$$

$$H = (\mathrm{i}k)^{-1} \int_0^l \mathrm{e}^{\mathrm{i}kx}\,(\mathrm{e}^{\mathrm{i}k_0 x}\sin\beta x)'\,\mathrm{d}x \int_0^x \mathrm{e}^{-\mathrm{i}k\xi}\,(\mathrm{e}^{\mathrm{i}k_0\xi}\sin\beta\xi)'\,\mathrm{d}\xi \tag{4.5a}$$

$$= \int_0^l \mathrm{e}^{\mathrm{i}kx}\,(\mathrm{e}^{\mathrm{i}k_0 x}\sin\beta x)'\,\mathrm{d}x\left[(\mathrm{i}k)^{-1}\mathrm{e}^{-\mathrm{i}kx}\,(\mathrm{e}^{\mathrm{i}k_0 x}\sin\beta x) + \int_0^x \mathrm{e}^{\mathrm{i}(k_0-k)\xi}\sin\beta\xi\,\mathrm{d}\xi\right] \tag{4.5b}$$

$$= \int_0^l e^{i(k_0+k)x}\,(ik_0 \sin\beta x + \beta\cos\beta x)\,dx \int_0^x e^{i(k_0-k)\xi} \sin\beta\xi\, d\xi \tag{4.5c}$$

$$= \beta^2[\beta^2-(k-k_0)^2]^{-1}\{\tfrac{1}{4}(ik_0)^{-1}(1-e^{2ik_0 l}) + ik[\beta^2-(k+k_0)^2]^{-1}[e^{i(k+k_0)l}\cos\beta l - 1]\}, \tag{4.5d}$$

and we have invoked $\sin\beta l = 0$ and $\cos 2\beta l = 1$. Substituting (4.5d) into (4.3), separating the propagated ($k = k_0$) and trapped ($k = i\ell$) terms, and invoking (4.2), we obtain

$$R_2 = 2i\varepsilon(k_0/\beta)\, R_1 + \tfrac{1}{2}\varepsilon^2(1-e^{2ik_0 l}) + R_{2\ell}, \tag{4.6}$$

where

$$R_{2\ell} = \tfrac{1}{2}\varepsilon^2(\beta^2/K_0)\sum_\ell K(i\ell)\left\{\frac{1-e^{2ik_0 l}}{\beta^2+(\ell+ik_0)^2} + \frac{4ik_0\ell[1-e^{-\ell l+ik_0 l}\cos\beta l]}{|\beta^2+(\ell+ik_0)^2|^2}\right\}. \tag{4.7a}$$

$$= \varepsilon^2(\beta^2/K_0)\, e^{i(k_0 l-\pi/2)}\, \mathrm{Im}\sum_\ell K(i\ell)\left[\frac{e^{ik_0 l}-e^{-\ell l}\cos\beta l}{\beta^2+(\ell+ik_0)^2}\right], \tag{4.7b}$$

and (4.7b) follows from (4.7a) through the identity

$$\frac{4ik_0\ell}{|\beta^2+(\ell+ik_0)^2|^2} = \frac{1}{\beta^2+(\ell-ik_0)^2} - \frac{1}{\beta^2+(\ell+ik_0)^2} \tag{4.8}$$

and the reality of

$$K(i\ell) = (\ell^2+\kappa^2)/[(\ell^2+\kappa^2)\,h_0-\kappa]. \tag{4.9}$$

The series in (4.7b) may be summed by invoking the rigid-lid approximations $\ell \simeq s\pi/h_0$ ($s = 1, 2, \cdots$) and $K(i\ell) \simeq 1/h_0$ for the trapped modes and neglecting the exponentially small terms to obtain

$$R_{2\ell} = \varepsilon^2(\beta^2/K_0)\, e^{i(k_0 l-\pi/2)}\, \mathrm{Im}\,(e^{ik_0 l}S), \tag{4.10a}$$

$$S \equiv \sum_\ell \frac{K(i\ell)}{\beta^2+(\ell+ik_0)^2} = \frac{1}{2i\pi\beta}\sum_{s=1}^{\infty}\left[\frac{1}{s+i(h_0/\pi)(k_0-\beta)} - \frac{1}{s+i(h_0/\pi)(k_0+\beta)}\right]$$
$$= (2i\pi\beta)^{-1}\{\psi[1+i(h_0/\pi)(k_0+\beta)] - \psi[1+i(h_0/\pi)(k_0-\beta)]\}, \tag{4.10b}$$

where ψ is the digamma function (Abramowitz & Stegun 1964, hereinafter referred to as AS, §6.3).

5. Bragg resonance

Bragg resonance occurs for $k_0 l = \tfrac{1}{2}\beta l = \tfrac{1}{2}m\pi$, for which (4.2) and (4.6) reduce to

$$R_1 = \tfrac{1}{4}m\pi\varepsilon, \quad R_2 = (\sin^2\tfrac{1}{2}m\pi + \tfrac{1}{4}im\pi)\,\varepsilon^2 + R_{2\ell}, \tag{5.1 a, b}$$

and the approximation (4.10) yields

$$R_{2\ell} = \tfrac{1}{2}i^{m-1}m\varepsilon^2(K_0 l)^{-1}\,\mathrm{Im}\left\{i^{m-1}\left[\psi\left(1+i(m+\tfrac{1}{2})\frac{h_0}{l}\right) - \psi\left(1-i(m-\tfrac{1}{2})\frac{h_0}{l}\right)\right]\right\} \tag{5.2a}$$

$$= i^{m-1}\varepsilon^2(\mu_0/\pi)\,\mathrm{Im}\{i^{m-1}[\psi(1+i\mu_+) - \psi(1-i\mu_-)]\}, \tag{5.2b}$$

where

$$\mu_0 \equiv k_0/K_0 = k_0 h_0 + \tfrac{1}{2}\sinh 2k_0 h_0, \quad \mu_\pm \equiv \left(\frac{2m\pm 1}{m\pi}\right)k_0 h_0. \tag{5.3 a, b}$$

Letting m be either odd or even and invoking AS, §6.3 (13) and (17), we obtain

$$R_{2\ell} = \tfrac{1}{2}\varepsilon^2\mu_0[\coth(\pi\mu_+) + \coth(\pi\mu_-) - (\pi\mu_+)^{-1} - (\pi\mu_-)^{-1}] \quad (m\ \text{odd}) \tag{5.4a}$$

m	1	4	8	20
b/h_0	$\frac{1}{3}$	0.320	0.320	0.160
$k_0 h_0$	$\frac{1}{2}$	0.491	0.491	0.982
ε	0.153	0.148	0.148	0.058
R_1 (5.1 a)	0.120	0.464	0.927	0.904
$R_{2\ell}$ (5.4)	0.0077	$O(10^{-4})$	$O(10^{-4})$	$O(10^{-4})$
R_2 (5.1 b)	0.031 + i.018	0.069i	0.138i	0.052i
$\lvert R_1+R_2 \rvert$	0.152	0.469	0.937	0.906
$\lvert R_1+R_2+R_3 \rvert$ (5.9)	0.155	0.446	0.746	0.670
tanh R_1	0.119	0.433	0.729	0.718
$\lvert R \rvert_{DH}$	—	0.45	0.72	0.68
$(k_0 a)_{DH}$	—	0.027	0.027	0.054

TABLE 1. Peak reflection coefficient for the sinusoidal patch (4.1), as calculated in §5 and measured by Davies & Heathershaw (1984) (DH).

or
$$R_{2\ell} = \mathrm{i}\varepsilon^2(\mu_0/\pi)(\mu_+^2-\mu_-^2)\sum_{n=1}^{\infty}\frac{n}{(n^2+\mu_+^2)(n^2+\mu_-^2)} \quad (\mu \text{ even}). \tag{5.4b}$$

It follows from (5.1) and (5.4) that if m is even R_2 is in quadrature with R_1 and therefore contributes only $O(\varepsilon^3)$ to $|R|$. But if m is odd R_2 has an in-phase (with R_1) component and contributes $O(\varepsilon^2)$ to $|R|$.

The second-order approximation described by (5.1)–(5.4) is adequate for $m\varepsilon$ 1. The simplest case is a half-wave bump, for which $m = 1$,

$$R_1 = \tfrac{1}{4}\pi\varepsilon, \quad R_2 = (1+\tfrac{1}{4}\mathrm{i}\pi)\,\varepsilon^2 + R_{2\ell}, \tag{5.5a, b}$$

and
$$R_{2\ell} = \tfrac{1}{2}\varepsilon^2\mu_0[\coth(3k_0 h_0)+\coth(k_0 h_0)-\tfrac{4}{3}(k_0 h_0)^{-1}]. \tag{5.6}$$

Adding (5.5 a) and (5.5 b), we place the resulting second-order approximation in the form

$$R = \tfrac{1}{4}\pi\varepsilon + \varepsilon^2\{1+\tfrac{1}{4}\mathrm{i}\pi+\tfrac{1}{2}[1+(2k_0 h_0)^{-1}\sinh 2k_0 h_0] \times [k_0 h_0(\coth 3k_0 h_0+\coth k_0 h_0)-\tfrac{4}{3}]\}. \tag{5.7}$$

But if $m\varepsilon = O(1)$ the second-order approximation is inadequate for the calculation of the peak (Bragg-resonant) reflection coefficient, and it is necessary to include the third-order contribution of the propagated mode. This calculation is sketched in the Appendix and yields

$$R_3 = \varepsilon^3[-\tfrac{1}{3}(\tfrac{1}{4}m\pi)^3+\tfrac{13}{8}\mathrm{i}(\tfrac{1}{4}m\pi)^2+O(\tfrac{1}{4}m\pi)], \tag{5.8}$$

in which the neglected terms are of the same order as the trapped-mode component $R_{3\ell}$. Adding the dominant parts of (5.1 a), (5.1 b) for m large and even, and (5.8), we obtain

$$R = R_1 - \tfrac{1}{3}R_1^3 + \mathrm{i}\varepsilon(R_1+\tfrac{13}{8}R_1^2). \tag{5.9}$$

The real part of (5.9) comprises the first two terms in the R_1 expansion of Mei's (1985) asymptotic ($\beta l \uparrow \infty$ with R_1 fixed) approximation ($x = 0$ in Mei's (3.24))

$$R \sim \tanh R_1, \tag{5.10}$$

but the imaginary part differs significantly from, Mei's M.

Numerical values of the above approximations for the half-wave bump and the experimental configurations of Davies & Heathershaw are given in table 1. The experimental values of $(k_0 a/\varepsilon)^2$ are small, as appears to be required for the neglect of nonlinearity (see last paragraph in §1), for $m = 4$ and 8, but not for $m = 20$; however, the coefficient of $k_0^2 a^2$ in the correction for nonlinearity typically is small, and Davies & Heathershaw's results do not appear to be amplitude dependent.

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Appendix. Reduction of R_3

The reduction of Q_3 and R_3 follows that of Q_2 and R_2 in §3 and yields

$$Q_3(x) = h_1(x)\,\partial_x \int_0^l G(x-\xi)\,Q_2'(\xi)\,\mathrm{d}\xi + \tfrac{1}{6}\mathrm{i}k_0^3\,h_1^3(x)\,\mathrm{e}^{\mathrm{i}k_0 x} \tag{A 1}$$

and

$$R_3 = -(\mathrm{i}k_0)^{-1} K_0 \int_0^l \mathrm{d}x \int_0^x G(x-\xi)\,[Q_1'(x)\,Q_2'(\xi) + Q_1'(\xi)\,Q_2'(x)]\,\mathrm{d}\xi + \tfrac{1}{6}\mathrm{i}k_0^3 K_0 \int_0^l h_1^3(x)\,\mathrm{e}^{2\mathrm{i}k_0 x}\,\mathrm{d}x. \tag{A 2}$$

We restrict further consideration to the Bragg-resonant sinusoidal patch, for which h_1 is given by (4.1) and $k_0 l = \frac{1}{2}\beta l = \frac{1}{2}m\pi$, and neglect the third-order contributions of the trapped modes. Substituting the k_0 component of G from (2.9*b*) into (3.3) and invoking $k_0 = \frac{1}{2}\beta$, we obtain

$$Q_2(x) = \tfrac{1}{2}\mathrm{i}b^2 K_0 \sin\beta x\, E'(x), \tag{A 3}$$

where

$$E(x) = \tfrac{1}{2}\mathrm{i}\beta(l-x)\,\mathrm{e}^{-\mathrm{i}k_0 x} - (1-\cos\beta x - \tfrac{1}{2}\mathrm{i}\sin\beta x)\,\mathrm{e}^{\mathrm{i}k_0 x}. \tag{A 4}$$

The corresponding approximation to R_3, obtained by integrating the terms in Q_1' and Q_2' by parts and substituting Q_1 and Q_2 from (3.1) and (A 3), is

$$R_3 = \tfrac{1}{16}(k_0 b)^3 K_0 l - 2(\mathrm{i}k_0)^{-1} K_0^2 \int_0^l Q_1(x)\,Q_2(x)\,\mathrm{d}x - K_0^2 \int_0^l \mathrm{e}^{\mathrm{i}k_0 x}\,\mathrm{d}x \int_0^x \mathrm{e}^{-\mathrm{i}k_0 \xi}\,[Q_1(x)\,Q_2(\xi) + Q_1(\xi)\,Q_2(x)]\,\mathrm{d}\xi. \tag{A 5a}$$

$$= \varepsilon^3[-\tfrac{1}{3}(\tfrac{1}{4}m\pi)^3 + (\tfrac{77}{64} + \tfrac{1}{8}\mu_0^2)(\tfrac{1}{4}m\pi) + \tfrac{13}{8}\mathrm{i}(\tfrac{1}{4}m\pi)^2 + \tfrac{1}{3}\mathrm{i}\sin^2(\tfrac{1}{2}m\pi)]. \tag{A 5b}$$

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